Table 1 Material properties used in the numerical solution

Material	E_1	E_2	ν_{12}	G_{12}
Graphite/epoxy	155 GPa	10 GPa	0.3	7 GPa

Table 2 Variation in maximum deflection relative to center deflection for AR = 0.6 plate

y axis location	27%	50%	73%	α
Theory	0.253 mm	0.246 mm	0.253 mm	2.8%
Experiment	0.245 mm	0.240 mm	0.255 mm	4.2%

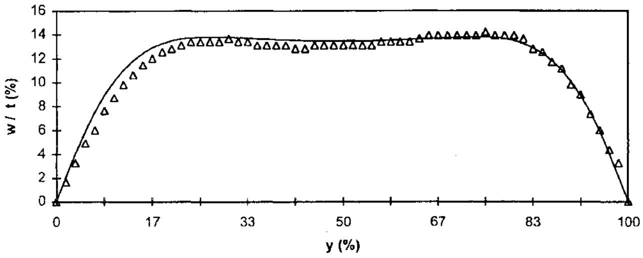

Fig. 3 Theoretical vs experimental plate deflection for AR = 0.62 graphite/epoxy unidirectional ply plate: ——, AR = 0.62 theory and △, AR = 0.62 experiment.

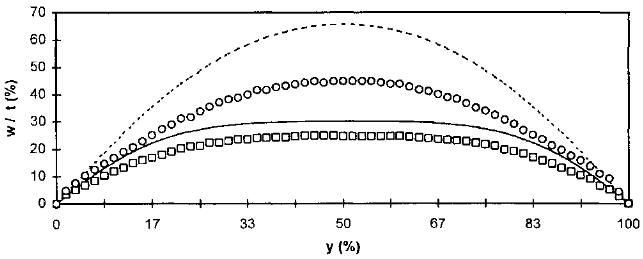

Fig. 4 Theoretical vs experimental plate deflection for AR = 1.00 and 1.88 graphite/epoxy unidirectional ply plate: ——, AR = 1.00 theory; ----, AR = 1.88 theory; □, AR = 1.00 experiment; and ○, AR = 1.88 experiment.

where s_1 and s_2 are the complex conjugate roots to the characteristic equation[1,2]

$$D_1 s^4 + 2D_3 s^2 + D_2 = 0$$

For a uniformly distributed load the particular solution is

$$w_p = (q_0/24D_2)(y^4 - 2by^3 + b^3 y)$$

with coefficients $a_n = (4q_0/\pi n)$ for $n = 1, 3, 5, \ldots$, and 0 for $n = 2, 4, 6, \ldots$.

The overall deflected plate shape and the deflected plate shape at $x = 0$ were calculated and plotted for the three AR tested. The material properties used in the numerical solution are shown in Table 1 (Ref. 1).

Results and Discussion

A comparison of the experimental data with the theoretical results given by the Levy solution was made both numerically and graphically. A variable α is defined as the variation in maximum deflection relative to the center deflection,[1]

$$\alpha = \frac{w_{\max} - w_{\text{center}}}{w_{\text{center}}} \times 100$$

where $w_{\max}$ is taken as the average local maxima of the plate.

Values for α are shown in Table 2 for the AR = 0.6 plate. The measured deflections were taken at the locations of theoretical local maxima because the experimental curve was not symmetric.

A graphical comparison of the theoretical and experimental results for the three AR tested is shown in Figs. 3 and 4. The shape of the theoretical plate deflection is close to the experimental data, which shows the predicted reverse curvature for AR < 1. The magnitude of theoretical deflection is conservative for the higher AR, which could not be explained by calculation or experimental measurement error. A brief study of the effect various parameters have on the numerical solution showed that it was possible to improve the accuracy of the maximum deflection results, while maintaining the proper curve shapes, by varying the material properties. The purpose of this short Note is to qualitatively validate theoretical predictions about the behavior of orthotropic plates; therefore the material property values given in Ref. 1 were used rather than experimentally obtained material properties. It is assumed that inaccurate material property values used in the numerical model, along with the inability to experimentally obtain theoretically perfect boundary conditions, is partly responsible for the deflection errors. However, the large difference between the theoretical and experimental deformations shown in Fig. 4 (AR = 1.00 and 1.88) is perhaps a result of the overprediction of deflection by small deformation plate theory. One may need large deformation theory to accurately predict deflections as large as one-half the plate thickness.

Conclusions

The experimental observations qualitatively confirm the analytically predicted behavior of orthotropic plates. This phenomenon of maximum deflection occurring at locations other than the center for orthotropic plates can be a significant factor in composite plate design.

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Eigenvector Derivatives with Repeated Eigenvalues Using Generalized Inverse Technique

Fu-Shang Wei*
Kaman Aerospace Corporation,
Bloomfield, Connecticut 06002
and
De-Wen Zhang†
Beijing Institute of Structure and Environment,
Beijing 100076, People's Republic of China

I. Introduction

DERIVATIVES of eigenvalues and eigenvectors are being used in the analysis for guidance in design modification and for improving analytical models in many technical fields.[1–5] The determination of eigenvalue derivatives can be a straightforward calculation, but the computation of eigenvector derivatives is very time consuming. The techniques of the calculation of eigenvector derivatives of a large complex eigensystem are problem dependent. The main reason is that the algebraic equations acquired upon differentiating the eigensystem relationships result in an undetermined system of equations. For the eigenvector derivative computation, the most simple approach is by finite difference method. Although the finite difference method is easy to implement into the computer program, in many cases the numerical perturbation step sizes and the selection of proper design parameters become even more expensive than other direct analytical methods.

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*Principal Engineer, Engineering and Development Department. Senior Member AIAA.

†Professor and Senior Research Engineer. Member AIAA.

When the solution of an eigensystem produces repeated eigenvalues, the computation of derivatives of the eigenvalues and eigenvectors becomes very complicated, as shown in Refs. 6–8. The complication is related to the fact that the eigenvectors of the repeated eigenvalues are not unique. There are an infinite number of linear combinations of the eigenvectors satisfying the eigensystem.

In Ref. 6, Ojalvo first extended Nelson's method and developed a solution to the eigenvector derivative with repeated eigenvalue, but Ojalvo's general solution does not possess the generality. To improve Ojalvo's method,[6] Mills-Curran[7] and Dailey[8] presented a complete method for calculating the general solution of the eigenvector derivatives with repeated eigenvalue problem. On solving the particular solution, Zhang and Wei's direct perturbation method[9] is simpler than Ojalvo's method,[6] since the direct perturbation method avoids the process exerting m constrains to the singular governing equation of the eigenvector derivative.

In this Note, the generalized inverse technique (GIT) is proposed for determining the particular solution of eigenvector derivative with and without repeated eigenvalues. The GIT can also avoid the process exerting one or m constrains to the governing equation. In addition, the GIT method does not contain the calculation of inverse of matrix. A numerical example shows that the precision of the GIT method is the same as that of Refs. 8 and 9.

II. Generalized Inverse

For any matrix $G(m \times n)$, there is a unique matrix $H(n \times m)$ satisfying Eqs. (1–4) and denoted as G^+:

$$GHG = G \tag{1}$$

$$HGH = H \tag{2}$$

$$(GH)^* = GH \tag{3}$$

$$(HG)^* = HG \tag{4}$$

where the asterisk represents the complex conjugate of a matrix and G^+ is known as the Moore–Penrose generalized inverse. But there are many generalized inverse matrices $H(n \times m)$ that satisfy just the first Penrose condition of Eq. (1). Note that matrix H is a generalized inverse of G and not the generalized inverse. The exception is when matrix G is nonsingular, in which case there is only one matrix H satisfying Eq. (1). The GIT has been applied to solve the linear structural systems and to compute eigenvector derivatives with distinct eigenvalues.[10,11]

A general linear system equation can be expressed as

$$Gx = b \tag{5}$$

where G is a matrix with any rank, x is an $(n \times 1)$ column vector, and b is an $(m \times 1)$ column vector. The general solution of Eq. (5) yields

$$x = G^+b + (I - G^+G)z \tag{6}$$

where $z(n \times 1)$ is any arbitrary vector. The second term on the right-hand side of Eq. (6) corresponds to a homogeneous solution of Eq. (5) because $(I - G^+G)z$ satisfies $Gx = 0$. That is, the column space of the homogeneous solution is contained in the null space of G. The column space of the particular solution of Eq. (5), G^+b, is perpendicular to the column space of the homogeneous solution. For this reason, G^+b is called the minimum norm least-square solution of Eq. (5).

III. Eigenvector Derivatives with Repeated Eigenvalues

It is well known that by solving the following subeigensystem[7,8] one can obtain the derivatives λ_j' $(j = 1, 2, \ldots, m)$ of m-multiple eigenvalue λ as well as the factor vector α_j:

$$[X^t(K' - \lambda M')X]\alpha_j = \lambda_j'\alpha_j \tag{7}$$

in which $X = [x_1, x_2, \ldots, x_m]$ is the eigenvector matrix of repeated root λ obtained directly by the eigenequation and α_j are the factor vector in linear combination $y_j = X\alpha_j$. Obviously, y_j $(j = 1, 2, \ldots, m)$ are also the eigenvectors of λ. The eigenvector derivatives with repeated frequencies can be determined by using the differentiation of the eigenequation with respect to a design parameter of interest:

$$Gy_j' = q_j \tag{8}$$

in which

$$G = K - \lambda M \tag{9}$$

$$q_j = -(K' - \lambda M' - \lambda_j' M)y_j \tag{10}$$

where K and M are real symmetric semipositive definite stiffness matrix and positive definite mass matrix, respectively, and K' and M' are the derivatives of K and M. From Eq. (6) we know that the solution y_j' of Eq. (8) can be computed providing the generalized inverse G^+ of G can be accomplished.

With m-multiple eigenvalue λ, the rank of matrix G is equal to $n - m$; thus we can establish a nonsingular matrix F:

$$F = G + M\left[\sum_{j=1}^{m} d_j\left(u_j u_j^t\right)\right]M \tag{11}$$

in which d_j is any nonzero scalar constant; u_j is a normalized eigenvector that can be taken as x_j or y_j. The inverse of F is

$$F^{-1} = G^+ + \sum_{j=1}^{m}\left(u_j u_j^t\right)/d_j \tag{12}$$

The nonsingular property of matrix F and the correctness of Eq. (12) are proved as follows.

For convenience, u_j is taken as x_j. Define the complete eigenpairs of the eigenequation as

$$\tilde{X} = [\Phi, X], \qquad \tilde{\Lambda} = \mathrm{diag}[\Omega, \lambda I] \tag{13}$$

Submatrix $\Phi \in R^{n,r}$ is built by the eigenvectors except for the eigenvectors x_j $(j = 1, 2, \ldots, m)$ of the repeated eigenvalue λ under consideration. The term Ω is a diagonal matrix constructed by the eigenvalues ω_i $(i = 1, 2, \ldots, r)$ corresponding to Φ, i.e., $\Omega = \mathrm{diag}[\omega_1, \omega_2, \ldots, \omega_r]$. Subscript r is equal to $n - m$. Let $\tilde{X}$ normalized eigenvectors. From $\tilde{X}^t M \tilde{X} = I$ we know $\tilde{X}^{-1} = \tilde{X}^t M$. Thus

$$
\begin{aligned}
G &= K - \lambda M \\
&= \tilde{X}^{-t}\tilde{\Lambda}\tilde{X}^{-1} - \lambda \tilde{X}^{-t}\tilde{X}^{-1} \\
&= M\tilde{X}\tilde{\Lambda}\tilde{X}^t M - \lambda M \tilde{X}\tilde{X}^t M \\
&= M\tilde{X}(\tilde{\Lambda} - \lambda I)\tilde{X}^t M \\
&= M[\Phi, X]\left(\begin{bmatrix}\Omega & 0 \\ 0 & \lambda I\end{bmatrix} - \begin{bmatrix}\lambda I & 0 \\ 0 & \lambda I\end{bmatrix}\right)\begin{bmatrix}\Phi^t \\ X^t\end{bmatrix} M
\end{aligned} \tag{14a}
$$

$$G = M[\Phi, X]\begin{bmatrix}\Omega - \lambda I & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\Phi^t \\ X^t\end{bmatrix} M \tag{14b}$$

Equation (14) shows that the rank of G with repeated root λ is equal to r; that is, G is a singular matrix. Equation (11) is rewritten as follows:

$$F = G + MXDX^tM \tag{15}$$

in which $D = \mathrm{diag}[d_1, d_2, \ldots, d_m]$. Equation (15) is rewritten again as

$$F = G + M[\Phi, X]\begin{bmatrix}0 & 0 \\ 0 & D\end{bmatrix}\begin{bmatrix}\Phi^t \\ X^t\end{bmatrix} M \tag{16}$$

Substituting Eq. (14) into Eq. (16) yields

$$F = M[\Phi, X]\left[\begin{array}{c:c} \Omega - \lambda I & 0 \\ \hdashline 0 & D \\ & (m, m) \end{array}\right]\begin{bmatrix}\Phi' \\ X'\end{bmatrix} M \tag{17}$$

Obviously, F is a nonsingular matrix, F^{-1} is given from Eq. (17):

$$F^{-1} = [\Phi, X]\left[\begin{array}{c:c} (\Omega - \lambda I)^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right]\begin{bmatrix}\Phi' \\ X'\end{bmatrix}$$

$$+ [\Phi, X]\left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & D^{-1} \end{array}\right]\begin{bmatrix}\Phi' \\ X'\end{bmatrix} \tag{18}$$

If we define the generalized inverse

$$(\tilde{\Lambda} - \lambda I)^{+} = \left[\begin{array}{c:c} (\Omega - \lambda I)^{-1} & 0 \\ \hdashline 0 & 0 \\ & (m, m) \end{array}\right] \tag{19}$$

then the generalized inverse G^{+} can be expressed from Eq. (14):

$$G^{+} = \tilde{X}(\tilde{\Lambda} - \lambda I)^{+}\tilde{X}' \tag{20a}$$

$$G^{+} = \Phi(\Omega - \lambda I)^{-1}\Phi' \tag{20b}$$

As earlier, Eq. (18) can be rewritten as

$$F^{-1} = G^{+} + XD^{-1}X'$$

$$= G^{+} + \sum_{j=1}^{m}\left(x_j x_j'\right)/d_j \tag{21}$$

Equation (21) is the same as Eq. (12) with u_j equal to x_j. Note that although Eq. (20b) can be employed to compute directly G^{+}, this formula is unsuitable for application. In addition, the generalized inverse G^{+} obtained by Eq. (12) can satisfy Moore–Penrose relations: $GG^{+}G = G$ and $G^{+}GG^{+} = G^{+}$ by using the formulas (14) and (20).

From Eq. (12) we obtain

$$G^{+} = F^{-1} - \sum_{j=1}^{m}\left(u_j u_j'\right)/d_j \tag{22}$$

After obtaining the generalized inverse G^{+} of G from Eq. (22), the general solution of y_j' is given in the basis of Eqs. (6) and (8):

$$y_j' = G^{+}q_j + (I - G^{+}G)z \tag{23}$$

Because the rank of G is of order $(n - m)$, this implies that the rank of $(I - G^{+}G)$ is of order m. Thus, there is

$$(I - G^{+}G)z = \sum_{j=1}^{m} c_j y_j \tag{24}$$

where c_j denotes any arbitrary scalar. Because the column space of G^{+} is orthogonal to y_i and G^{+} is unique, one can determine c_j from any meaningful constraints, i.e., the norm constraint used in Ref. 6 and the differentiation of the orthogonalization equations used in Refs. 7 and 8. The latter is utilized in this Note to compare with numerical results of Ref. 8.

IV. Algorithm Applied to Engineering

To obtain G^{+}, the inverse F^{-1} of matrix F must be computed. For large and complex space structures, the computation of the inverse matrix F^{-1} is difficult. To avoid the calculation of the inverse F^{-1}, an improved algorithm is proposed here. Define the particular solution of y_j' as

$$v_j = G^{+}q_j \tag{25}$$

Now an interim solution (also called a perturbed solution) $\tilde{v}_j$ of v_j is first found by using F in Eq. (26):

$$F\tilde{v}_j = q_j \tag{26}$$

Equation (26) is rewritten as

$$\tilde{v}_j = F^{-1}q_j \tag{27}$$

Substituting Eq. (12) into Eq. (27) yields

$$\tilde{v}_j = G^{+}q_j + UD^{-1}U'q_j$$

$$= v_j + UD^{-1}U'q_j \tag{28}$$

in which $U = [u_1, u_2, \ldots, u_m]$. From Eq. (28), the desired v_j is obtained in Eq. (29):

$$v_j = \tilde{v}_j - UD^{-1}U'q_j, \qquad j = 1, 2, \ldots, m \tag{29}$$

Clearly, the algorithm in this section avoids the calculation of F^{-1} in the process for solving v_j, since after the interim solution $\tilde{v}_j$ is obtained by solving a set of linear algebraic equations (26), v_j is given by Eq. (29). The algorithm in this section is that solving Eq. (26) is substituted for the calculation of F^{-1}.

V. Relationship Between the GIT and Direct Perturbed Method

In this section, we will discuss the relationship between the method in this Note and the direct perturbation (DP) method proposed in Ref. 9. The idea of the DP method is that the repeated eigenvalue λ under consideration in Eq. (8) is slightly perturbed to obtain Eq. (30):

$$\bar{F}y_j' = q_j \tag{30}$$

where

$$\bar{F} = K - (\lambda + \varepsilon)M \tag{31a}$$

$$\bar{F} = G - \varepsilon M \tag{31b}$$

is nonsingular[9] and ε is a small nonzero scalar constant, also called the perturbation constant. Equation (31b) can be rewritten as

$$\bar{F} = M\tilde{X}(\tilde{\Lambda} - \lambda I)\tilde{X}'M - \varepsilon M\tilde{X}\tilde{X}'M \tag{32}$$

Embedding Eq. (13) into Eq. (32) yields

$$\bar{F} = M[\Phi, X]\left[\begin{array}{c:c} \Omega - (\lambda + \varepsilon)I & 0 \\ \hdashline 0 & -\varepsilon I \end{array}\right]\begin{bmatrix}\Phi' \\ X'\end{bmatrix} M \tag{33}$$

The inverse of $\bar{F}$ is

$$\bar{F}^{-1} = [\Phi, X]\left\{\left[\begin{array}{c:c} [\Omega - \lambda(+\varepsilon)I]^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] - \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & (1/\varepsilon)I \end{array}\right]\right\}\begin{bmatrix}\Phi' \\ X'\end{bmatrix} \tag{34}$$

Clearly, when ε is a small scalar, the first term in Eq. (34) is approaching G^{+}. Thus Eq. (34) becomes

$$\bar{F}^{-1} = G^{+} - (1/\varepsilon)XX' \tag{35}$$

But the second term in Eq. (35) must not be small. Multiplying both sides of Eq. (34) by $X'M$ gives

$$X'M\bar{F}^{-1} = -(1/\varepsilon)X' \tag{36}$$

Inserting Eq. (36) into Eq. (35) yields

$$\bar{F}^{-1} = G^{+} + XX'M\bar{F}^{-1} \tag{37}$$

i.e.,

$$G^{+} = (I - XX'M)\bar{F}^{-1} \tag{38}$$

Similarly, to avoid the calculation of $\bar{F}^{-1}$ in Eq. (38), an interim solution $\tilde{v}_j$ is first computed in Eq. (39):

$$\bar{F}\tilde{v}_j = q_j \tag{39}$$

i.e.,

$$\tilde{v}_j = \bar{F}^{-1}q_j \tag{40}$$

Embedding Eq. (37) into Eq. (40) gives

$$v_j = (I - XX'M)\tilde{v}_j \tag{41}$$

Both Eqs. (39) and (41) are two principal formulas, as shown in Ref. 9. It is concluded that the generalized inverse technique described in this Note is also one of the DP methods as shown in Ref. 9. This is known by the fact that $MUDU'M$ (or $MXDX'M$) in Eq. (11) is considered as the perturbation term of DP method in Ref. 9.

VI. Numerical Results

The particular solution v_j of a numerical example shown in Ref. 8 is computed by using the GIT method in this paper and the scalar constant c_j in the general solution $y'_j = v_j + Yc_j$ of eigenvector derivatives is given by the procedure stated in Refs. 7 and 8, in which $Y = [y_1, y_2, \dots, y_m]$. The results (general solution y'_j) are the same as those of Refs. 8 and 9. In addition, to address the effect of different values of d_j, we select $d_j = d = 0.000001$, 1.0, and 100,000 ($j = 1, 2, \dots, m$), in which $d_j = d$ means that all d_j ($j = 1, 2, \dots, m$) are taken as a uniform value d. It is found that for very different d, the obtained solution y'_j is the same and that the particular solution v_j equals to the general solution y'_j. The former tells us that this method is numerically stable. The latter tells us that the particular solution v_j obtained by this method possibly happens to be just the general solution y'_j. Thus, it is very meaningful to propose a simple criterion by which we can know whether the v_j given by this method is y'_j. Finally, the results with $u_j = y_j$ and $u_j = x_j$ are the same.

VII. Conclusions

Based on technical analysis and numerical results, a new approach for determining the eigenvector derivatives with repeated roots has been successfully developed. This method can be applied to the calculation of eigenvector derivatives with and without repeated roots and avoids the process exerting one or m constrains to the governing equation of the eigenvector derivative. This method requires only the knowledge of the eigenvalue and its associated eigenvector under consideration. This method is numerically stable and requires no numerical convergence checks when incorporated into the existing computer program.

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Distributed Modeling and Actuator Location for Piezoelectric Control Systems

M. Sunar*
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
and
S. S. Rao†
Purdue University, West Lafayette, Indiana 47907-1288

Introduction

THE piezoelectric materials have been used extensively in disturbance sensing and control. Since the piezoelectric materials are distributed (not discrete) in nature, an accurate sensing and control of structures can be achieved by piezoelectric sensors and actuators.

The finite element method is a powerful technique widely used in many modern engineering design and analysis problems. Since the governing equations of piezoelectric media are complex in general, the finite element modeling suitable for the thin piezoelectric media becomes important.[1] The problem of sensor and actuator placement exists with piezoelectric sensors and actuators if they are discretely placed on various members. Hence, the issue of the piezoelectric sensor and actuator problem has been addressed[2] to see the effect of the placement on the control efficiency.

In this work, two-dimensional thin rectangular elements with pseudointernal degrees of freedom (DOF) are used to develop the finite element model of piezoelectric materials. The internal DOF are for the better representation of bending moments generated by the feedback piezoelectric voltage and are condensed to the physical nodal DOF using the Guyan reduction technique. The control efficiency effect of the location of the piezoelectric actuator pair on the beam is studied.

Finite Element Modeling of Piezoelectric Media

The quasistatic piezoelectric equations are given as[3]

$$\begin{aligned} \boldsymbol{T} &= c\boldsymbol{S} - e\boldsymbol{E} \\ \boldsymbol{D} &= e^{T}\boldsymbol{S} + \varepsilon\boldsymbol{E} \end{aligned} \tag{1}$$

where $\boldsymbol{T}, \boldsymbol{S}, \boldsymbol{E}$, and $\boldsymbol{D}$ are the vectors of stress, strain, electric field, and charge per unit area and c, e, and ε are the matrices of

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*Assistant Professor, Mechanical Engineering Department.

†Professor, School of Mechanical Engineering, 1288 Mechanical Engineering Building. Member AIAA.